

One-dimensional compression of a collisionless gas

By G. A. BIRD

Department of Mechanics of Fluids, University of Manchester

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A highly rarefied gas, initially in equilibrium, is compressed by an infinite plane piston. The resulting flow is considered when the gas is unbounded and also when the gas is bounded by a second stationary wall. In the first case, the density ahead of the piston is analysed for specularly and diffusely reflecting surfaces. In the second case, the average temperature of the gas is found as a function of the wall separation for specularly reflecting surfaces. This (non-equilibrium) temperature is compared with that generated in the corresponding continuum flow which involves multiple shock wave reflexion. It is shown that, for a given average density gradient, the free molecule temperatures are very much higher than the continuum temperatures.

1. Introduction

The solution of the one-dimensional piston problem for a gas which may be treated as a continuum is well known. A shock wave is formed ahead of a piston moving into the gas and, if the flow is bounded by a stationary wall, the shock is reflected backwards and forwards between the walls. However, at such short times after the start of the motion that the disturbance has propagated only a few mean free paths, or when the wall separation becomes of the order of the mean free path, the continuum solution will be in error. Because of the extreme difficulty in analysing flows with a typical dimension of the order of the mean free path, it is useful to obtain an indication of the direction and magnitude of the rarefied gas effects by analysing the limiting case in which collisions between the molecules may be neglected.

Since the flow in this free molecule limit is determined by collisions between the molecules and the boundaries, the nature of the interaction between the molecule and the boundary is all important. However, the details of the reflexion process vary in an often unknown manner with the nature of the surface and the velocity of the incident molecule, and it is necessary to treat idealized models. The analysis for the unbounded gas in §2 is carried out for the two classical models of specular and diffuse reflexion. Specular reflexion is an elastic process in which the velocity component normal to the wall is reversed and that parallel to the wall is retained, while a diffusely reflected molecule has its temperature adjusted towards that of the surface and is re-emitted in a random direction with a Maxwellian speed distribution. In the case of the bounded gas, the emphasis is on the temperature attained by the gas and only specular reflexion is considered.

2. Unbounded gas

The infinite plane piston is initially at rest and, at time $t = 0$, it acquires a velocity U in the direction normal to its face. The gas in front of the piston is initially stationary and in thermal equilibrium at the temperature T . It is convenient to choose rectangular co-ordinates moving with the piston and with the x -direction normal to the plane of the piston. In this frame of reference, the problem is equivalent to a plane wall being instantaneously inserted into a uniform stream of velocity $-U$. The flow upstream of the wall corresponds to the compression side of the piston, and this will be affected by the molecules reflected from the wall and also by the absence of the molecules which would previously come from positions downstream of the wall.

Now, the number flux of molecules in the x -direction with x , y and z velocity components in the range u to $u + du$, v to $v + dv$ and w to $w + dw$ is

$$dN = n f(u, v, w) u du dv dw, \quad (1)$$

where $f(u, v, w)$ is the velocity distribution function and n is the number density of the gas. Therefore, for specular reflexion, the number flux of molecules proceeding upstream with x velocity components in the range du due to the presence of the wall is

$$\begin{aligned} dN &= \frac{n_a \beta^3}{\pi^{\frac{3}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\exp\{-\beta^2[(-u+U)^2 + v^2 + w^2]\} \\ &\quad - \exp\{-\beta^2[(u+U)^2 + v^2 + w^2]\}] u du dv dw, \\ &= \frac{n_a \beta}{\pi^{\frac{1}{2}}} [\exp\{-\beta^2(U-u)^2\} - \exp\{-\beta^2(U+u)^2\}] u du. \end{aligned} \quad (2)$$

Here, n_a is the number density of the ambient molecules and $\beta (= (2RT)^{-\frac{1}{2}})$ is the reciprocal of the most probable thermal speed of these molecules. After time t , this stream of molecules contributes to the number density at a distance x from the piston if $x \leq ut$ (or $u \geq x/t$). Therefore, the additional number density is

$$n - n_a = \frac{n_a \beta}{\pi^{\frac{1}{2}}} \int_{x/t}^{\infty} [\exp\{-\beta^2(U-u)^2\} - \exp\{-\beta^2(U+u)^2\}] du,$$

and the ratio of the total to the ambient density is

$$n/n_a = 1 + \frac{1}{2} [\operatorname{erf}(\beta x/t + s) - \operatorname{erf}(\beta x/t - s)]. \quad (3)$$

The molecular speed ratio $s (= U\beta)$ is the ratio of the piston speed to the most probable thermal speed of the ambient molecules. The number density ratio at the piston face after it has been set in motion is obtained by setting $x = 0$ in (3). That is

$$n/n_a = 1 + \operatorname{erf} s. \quad (4)$$

Some typical results for the case of specular reflexion are shown in figure 1.

The corresponding result for inviscid continuum flow is a shock wave of constant strength propagating into the undisturbed gas (see, for example,

Courant & Friedrichs 1948). Table 1 gives a comparison of the density ratio across this shock wave with the density ratio at the face of a specularly reflecting piston.

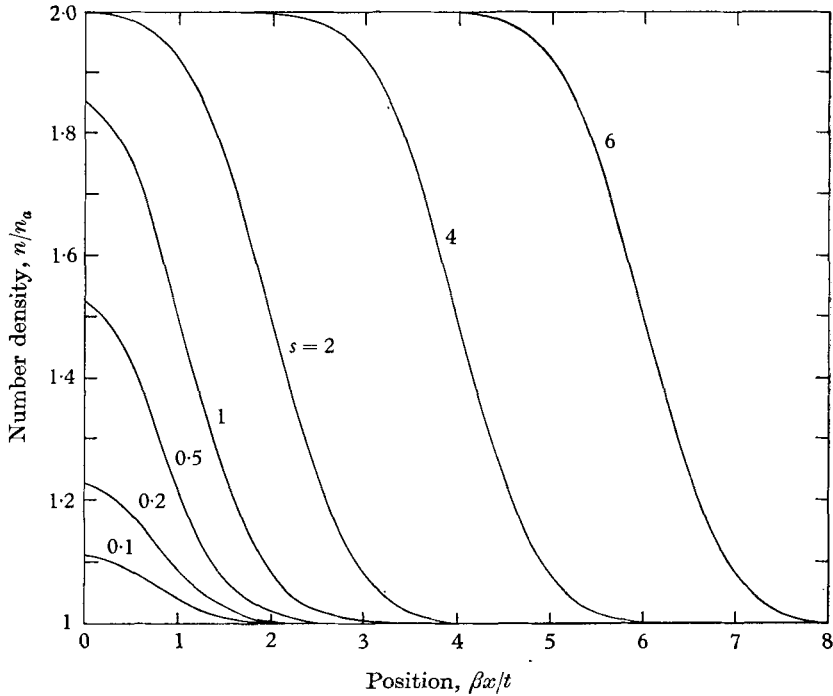


FIGURE 1. Density distribution ahead of specularly reflecting piston.

Molecular speed ratio s	Density ratio, n/n_a			
	Collisionless compression	Continuum compression		
		$\gamma = 1$	$\gamma = \frac{5}{3}$	$\gamma = 3$
0.1	1.1125	1.1518	1.1134	1.0814
0.2	1.2227	1.3257	1.2337	1.1612
0.5	1.5205	2.0000	1.6210	1.3780
1	1.8427	3.7231	2.2546	1.6325
2	1.9953	9.8990	3.1070	1.8528
4	2.0000	33.971	3.6931	1.9562
8	2.0000	130.0	3.9152	1.9885

TABLE 1

In the continuum flow, there is a transfer of energy between the molecular degrees of freedom and the result depends on the specific heat ratio γ of the gas. The value of γ has comparatively little effect at small values of s , and the continuum density ratios for a monatomic gas ($\gamma = \frac{5}{3}$) then show a remarkable agreement with the collisionless values. Since the disturbance in the collisionless flow is confined to one velocity component and therefore to one molecular degree

of freedom, the collisionless result for the larger values of s bears some relation to that given by continuum theory with $\gamma = 3$. For shock waves of moderate strength, a viscous theory for the structure of the wave during its formation has been given by Lighthill (1956).

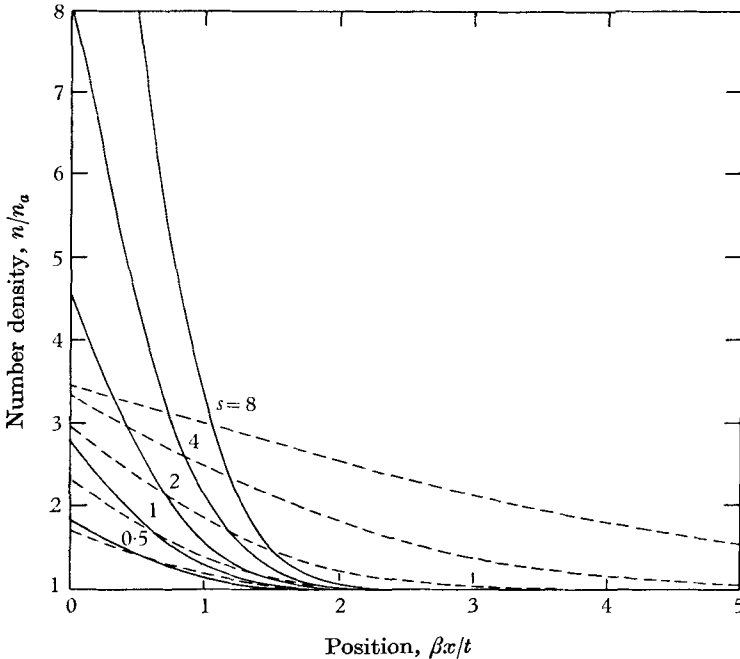


FIGURE 2. Density distribution ahead of diffusely reflecting piston. —, $T' = T$; ---, insulated piston.

In the case of diffuse reflexion, the second term in equation (2), which allows for the molecules which would have come from a point downstream of the piston face, remains unchanged. The new term for the reflected molecules is easily written down as the reflected molecules may be regarded as effusing from a fictitious gas of number density n_f in Maxwellian equilibrium at a temperature T' on the reverse side of the surface. That is

$$dN = \pi^{-\frac{1}{2}} [n_f \beta' \exp \{-\beta'^2 u^2\} - n_a \beta \exp \{-\beta^2 (U + u)^2\}] u \, du, \tag{5}$$

where $\beta' = (2RT')^{-\frac{1}{2}}$. The number density n_f is given by the condition that the number flux of molecules leaving the surface must equal the number striking it. That is

$$n_f \beta' \int_0^\infty \exp \{-\beta'^2 u^2\} u \, du = n_a \beta \int_{-\infty}^0 \exp \{-\beta^2 (u + U)^2\} (-u) \, du,$$

or
$$n_f = n_a (\beta' / \beta) [\exp(-s^2) + \pi^{\frac{1}{2}} s (1 + \operatorname{erf} s)]. \tag{6}$$

When (6) is substituted into (5) and a similar process to that for specular reflexion is carried out, the number density ratio for diffuse reflexion is

$$n/n_a = 1 + \frac{1}{2} (T/T')^{\frac{1}{2}} [\exp(-s^2) + \pi^{\frac{1}{2}} s (1 + \operatorname{erf} s)] \times \{1 - \operatorname{erf} [(T/T')^{\frac{1}{2}} (\beta x/t)]\} - \frac{1}{2} [1 - \operatorname{erf} (\beta x/t + s)]. \tag{7}$$

Figure 2 shows results for the case in which the temperature T' of the reflected molecules is equal to ambient temperature T , and also for the case in which it is equal to the recovery temperature of an insulated piston. The recovery temperature is given by Schaaf & Chambré (1958) as

$$T'/T = \frac{1}{4}\{2s^2 + 5 - [1 + \pi^{\frac{1}{2}}s(1 + \operatorname{erf} s) \exp s^2]^{-1}\}. \quad (8)$$

For large molecular speed ratios and finite time, the number density at the piston face is

$$n/n_a = 1 + \pi^{\frac{1}{2}}(T/T')^{\frac{1}{2}}s. \quad (9)$$

It therefore increases with s when the piston temperature is equal to the ambient temperature, but is limited to $1 + (2\pi)^{\frac{1}{2}}$ for the insulated piston.

A feature of the collisionless compression is that the whole flow is affected by the reflexion process at the wall and, for diffuse reflexion, by the temperature of the wall. The corresponding continuum flow is affected only by heat transfer at the wall, and these effects are confined to a thermal boundary-layer region adjacent to the wall.

3. Bounded gas

The configuration is shown in the distance-time plane in figure 3, the origin being chosen on the fixed wall at the instant the piston starts moving with velocity $-U$. A convenient measure of time is the ratio δ of the distance d travelled by the piston to the initial separation d_0 of the piston and fixed wall.

A number of molecular paths are shown in figure 3. These are all for molecules which are moving, at time $t = 0$, with the same velocity component u in the x -direction. It is seen that, at the time at which the piston reaches A , molecules which initially lie between x_{-2} and x_{-1} have suffered one collision with the piston after having first collided with the fixed wall, those between x_{-1} and x_0 have not suffered a collision with the piston, those between x_0 and x_1 have suffered one collision and those between x_1 and x_2 two collisions. In general, molecules which initially have a velocity component u and lie between x_{n-1} and x_n have suffered $|n|$ collisions with the piston. If x_{n-1} or x_n lie outside the range 0 to d_0 , the limit on the range of x for molecules with the particular number of collisions must be replaced by 0 or d_0 as appropriate. For a given n between $-\infty$ and ∞ , the magnitude of the x velocity component is

$$|2nU + u|,$$

a negative n indicating that the molecule makes its first collision with the fixed wall. Since the reflexion is specular, the other velocity components are not affected by the collisions.

If $t = d/U$ is the time corresponding to the piston reaching the point A in figure 3, simple geometrical considerations show that

$$d_0 - Ut = x_0 + ut,$$

or

$$x_0/d_0 + (1 - \delta) - \delta(u/U).$$

Similarly, if t' is the time at which the molecule starting at x_1 strikes the moving wall, x_1 is given by the solution of the two equations

$$d_0 - Ut' = x_1 + ut',$$

and

$$2d_0 - d = Ut' + (u + 2U)(t - t'),$$

i.e.

$$x_1/d_0 = 3(1 - \delta) - \delta(u/U).$$

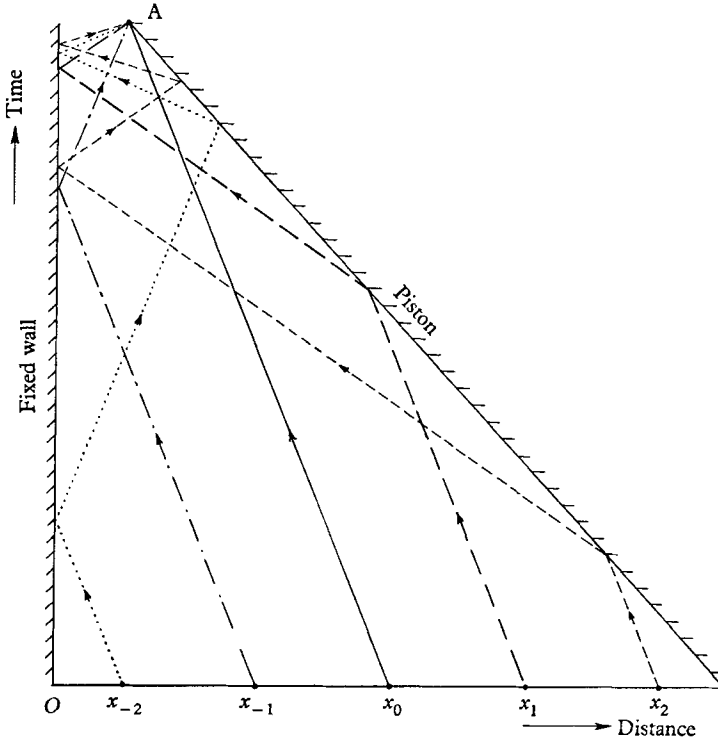


FIGURE 3. Typical molecular paths during the compression of a bounded gas.

The process may be repeated indefinitely and it is seen that

$$\left. \begin{aligned} x_n/d_0 &= (2n + 1)(1 - \delta) - \delta(u/U), \\ (x_n - x_{n-1})/d_0 &= 2(1 - \delta). \end{aligned} \right\} \quad (10)$$

and

Therefore, when $\delta \geq \frac{1}{2}$ so that $2(1 - \delta) \leq 1$, the fraction of molecules with x velocity component u which have suffered collisions with the piston is

$$\begin{aligned} &0 && \text{if } x_n < 0, \\ &x_n/d_0 && \text{if } x_n > 0 \quad \text{and } x_{n-1} < 0, \\ &2(1 - \delta) && \text{if } x_n < d_0 \quad \text{and } x_{n-1} > 0, \\ &1 - (x_{n-1}/d_0) && \text{if } x_n > d_0 \quad \text{and } x_{n-1} < d_0, \\ &0 && \text{if } x_{n-1} > d_0, \end{aligned}$$

where n may have any value between $-\infty$ and $+\infty$. Similarly, when $\delta \leq \frac{1}{2}$ the fraction is

$$\begin{aligned} & 0 \quad \text{if} \quad x_n < 0, \\ & x_n/d_0 \quad \text{if} \quad d_0 > x_n > 0, \\ & 1 \quad \text{if} \quad x_n > d_0 \quad \text{and} \quad x_{n-1} < 0, \\ & 1 - (x_{n-1}/d_0) \quad \text{if} \quad d_0 > x_{n-1} > 0, \\ & 0 \quad \text{if} \quad x_{n-1} > d_0. \end{aligned}$$

To treat one case in detail, consider the contribution ΔT to the temperature at a particular $\delta (> \frac{1}{2})$ of the molecules such that $x_n > 0$ and $x_{n-1} < 0$. These limits fix the range of u as from $(2n-1)(\delta^{-1}-1)U$ to $(2n+1)(\delta^{-1}-1)U$ and the fraction is $(2n+1)(1-\delta) - \delta u/U$. Therefore,

$$\begin{aligned} 3R\Delta T = \sum_{n=-\infty}^{\infty} \frac{\beta^3}{\pi^{\frac{3}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{(2n-1)(\delta^{-1}-1)U}^{(2n+1)(\delta^{-1}-1)U} \exp\{-\beta^2(u^2+v^2+w^2)\} \\ \times [(2nU+u)^2+v^2+w^2][(2n+1)(1-\delta) - \delta u/U] du dv dw. \end{aligned}$$

That is

$$\begin{aligned} \Delta \left(\frac{T}{T_0} \right) = \frac{2}{3\pi^{\frac{3}{2}}} \sum_{n=-\infty}^{\infty} \int_{(2n-1)(\delta^{-1}-1)s}^{(2n+1)(\delta^{-1}-1)s} \exp\{-(u\beta)^2\} [(2ns+u\beta)^2+1] \\ \times [(2n+1)(1-\delta) - \delta u\beta/s] d(u\beta), \end{aligned}$$

where T_0 is the equilibrium temperature of the gas before the piston starts moving. This integral is readily evaluated and, when the contributions of the other fractions are included, the final result for both $\delta > \frac{1}{2}$ and $\delta < \frac{1}{2}$ is

$$\begin{aligned} \frac{T}{T_0} = \frac{\delta}{3} \sum_{n=-\infty}^{\infty} \left[\frac{8n^2s^2+3}{2s} (\theta \operatorname{erf} \theta - \phi \operatorname{erf} \phi - \mu \operatorname{erf} \mu + \nu \operatorname{erf} \nu) \right. \\ \left. - 2n(\operatorname{erf} \theta - \operatorname{erf} \phi - \operatorname{erf} \mu + \operatorname{erf} \nu) + \frac{2(2n^2s^2+1)}{\pi^{\frac{1}{2}}s} (e^{-\theta^2} - e^{-\phi^2} - e^{-\mu^2} + e^{-\nu^2}) \right], \end{aligned} \tag{11}$$

where

$$\theta = (2n+1)(\delta^{-1}-1)s,$$

$$\phi = (2n-1)(\delta^{-1}-1)s,$$

$$\mu = [2n(\delta^{-1}-1)-1]s,$$

and

$$\nu = [2(n-1)(\delta^{-1}-1)-1]s.$$

Since $(\theta - \phi - \mu + \nu) = 0$, the terms in the series tend to zero as $|n| \rightarrow \infty$.

For very large molecular speed ratios, the initial thermal velocities are negligible and it may be assumed that only those molecules which have suffered at least one collision with the piston contribute to the temperature. It can be seen from equation 10 (or a diagram similar to figure 3 with the molecular paths initially drawn vertical) that a fraction $1-\delta$ of the molecules will not have suffered a collision, and fractions $2(1-\delta)$ will have suffered 1, 2, 3, ..., n collisions, until the sum of all the fractions exceeds unity, when the final fraction must be reduced to $1 - (2n+1)(1-\delta)$. Therefore, since the speed of the molecule increases by $2U$ with each collision, the temperature is given by

$$3RT = 8(1-\delta) \sum_{n=0}^m n^2U^2 + 4[1 - (2m+1)(1-\delta)](m+1)^2U^2, \tag{12}$$

where m is the integer below $\delta[2(1-\delta)]^{-1}$. If the molecular speed ratio is introduced, this may be written as a temperature ratio, i.e.

$$\frac{T}{T_0} = \frac{8s^2}{3} \left\{ 2(1-\delta) \sum_{n=0}^m n^2 + [1 - (2m+1)(1-\delta)](m+1)^2 \right\}. \quad (13)$$

The ratio of the average density of the gas to its initial density at $t = 0$ is equal to

$$\rho/\rho_0 = 1/(1-\delta),$$

and figure 4 shows some curves of temperature ratio against density ratio for various values of s . For $s = 10$ and 100, the results given by equation (13) are indistinguishable from those given by the exact equation (11) and show that the 'kinks' in these curves occur when the number of the successive multiple collisions suddenly becomes appreciable.

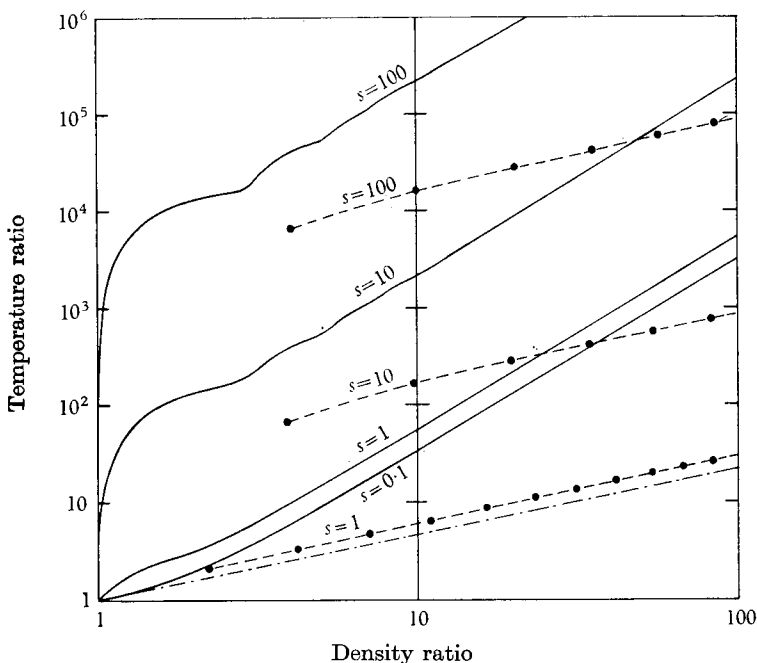


FIGURE 4. Temperature-density behaviour of a gas undergoing one-dimensional compression. —, Collisionless compression; - - -, shock compression ($\gamma = \frac{5}{3}$); - · - · -, adiabatic compression (and $s = 0.1$ shock compression).

Curves are also shown for the corresponding continuum flow which involves multiple shock reflexion (Evans & Evans 1956). These are, of course, for a monatomic gas (specific heat ratio $\gamma = \frac{5}{3}$) and the successive shock wave reflexions are indicated by the heavy dots on the curves. For $s = 0.1$, the compression is almost isentropic, i.e.

$$T/T_0 = (\rho/\rho_0)^{\gamma-1}. \quad (14)$$

For larger values of s , most of the entropy increase is across the first shock and the slope of the curves approaches the isentropic value.

The collisionless compression curves always lie above the corresponding shock compression curves and settle down to the slope

$$T/T_0 = (\rho/\rho_0)^2.$$

This is as expected, since the collisionless compression adds energy to only one degree of freedom so that the gas ultimately behaves as a perfect gas with $\gamma = 3$.

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